

Exercise 9

By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using inequality (8), Sec. 4, show that if z lies on the circle $|z| = 2$, then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}.$$

Solution

$$\begin{aligned} \left| \frac{1}{z^4 - 4z^2 + 3} \right| &= \frac{1}{|z^4 - 4z^2 + 3|} \\ &= \frac{1}{|(z^2 - 1)(z^2 - 3)|} \\ &= \frac{1}{|z^2 - 1||z^2 - 3|} \\ &\leq \frac{1}{||z^2| - |1|||z^2| - |3||} \\ &= \frac{1}{||z|^2 - 1||z|^2 - 3|} \\ &= \frac{1}{|2^2 - 1||2^2 - 3|} \\ &= \frac{1}{(3)(1)} \\ &= \frac{1}{3} \end{aligned}$$

Therefore, if $|z| = 2$, then

$$\left| \frac{1}{z^4 - 4z^2 + 3} \right| \leq \frac{1}{3}.$$